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B.M.S. COLLEGE FOR WOMEN

BENGALURU -560004

III SEMESTER END EXAMINATION – APRIL - 2024

M.Sc – MATHEMATICS - LINEAR ALGEBRA
(CBCS Scheme – F+R)

Course Code: MM301T **QP Code: 13001**
Duration: 3 Hours **Max. Marks: 70**

Instructions: 1) All questions carry equal marks.
2) Answer any five full questions.

1. a) If V is a finite dimensional vector space over F, then prove that $T \in A_F(V)$ is invertible if and only if the constant term of the minimal polynomial of T is non-zero.
   b) If V is a finite dimensional vector space over F, then prove that $T \in A_F(V)$ is regular if and only if T maps V onto itself.
   c) Define regular and singular transformation. If V is a finite dimensional vector space over F and if $T \in A_F(V)$ is regular, then prove that T^{-1} has a polynomial expression in T over F.
   (5+5+4)

2. a) Define characteristic root of a linear transformation. Prove that the non-zero characteristic vectors belonging to distinct characteristic roots are linearly independent.
   b) If $\lambda \in$ F is a characteristic root of $T \in A_F(V)$, then for any $q(x) \in F[x]$, prove that $q(\lambda)$ is a characteristic root of $q(T)$.
   c) Let V be the vector space of all polynomials of degree 3 or less over F. In V, define T by $(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3)T = \alpha_0 + \alpha_1(1+x) + \alpha_2(1+x^2) + \alpha_3(1+x^3)$.
   Compute the matrix of T in the following bases
   (i) $1, x, x^2, x^3$; let this matrix be A.
   (ii) $1, 1+x, 1+x^2, 1+x^3$; let this matrix be B.
   (iii) Find C such that $B = CAC^{-1}$. **(5+4+5)**

3. a) Prove that the double dual of V is isomorphic to V.
   b) Define coordinate vector $[x]_B$ of x relative to a basis B. Let $B = \{v_1, v_2, \cdots, v_n\}$ be a basis for the vector space V. Prove that the coordinate mapping $x \to [x]_B$ is a one to one linear transformation from V to R^n.
   c) Define the change of coordinates matrix from a basis B to C. Prove that $\left(\underset{C \leftarrow B}{P}\right)^{-1} = \left(\underset{B \leftarrow C}{P}\right)$. Illustrate it for the basis $B = \left\{\begin{bmatrix}7\\5\end{bmatrix}, \begin{bmatrix}-3\\-1\end{bmatrix}\right\}$ and $C = \left\{\begin{bmatrix}1\\5\end{bmatrix}, \begin{bmatrix}-2\\2\end{bmatrix}\right\}$.
   (6+4+4)

4. a) State and prove Cayley Hamilton theorem.
b) If $T \in A_F(V)$ has all its characteristics roots in F, then prove that there exists a basis of V in which the matrix of T is triangular.
c) Let $T \in A_F(V)$ and V_1 be a n_1 – dimensional subspace of V spanned by $\{v, T(v), \ldots T^{n_1-1}(v)\}$ where $v \neq 0$. If $u \in V_1$ is such that $T^{n_1-k}(u) = 0, 0 \leq k \leq n_1$, then prove that $u = T^k(u_0)$ for some $u_0 \in V_1$. **(4+6+4)**

5. a) Prove that two nilpotent transformations are similar if and only if they have the same invariants.
b) If $T \in A_F(V)$ has a minimal polynomial $p(x) = \gamma_0 + \gamma_1 x + \cdots + \gamma_{r-1}x^{r-1} + x^r$ over F. Suppose V as a module is a cyclic module relative to T. Then prove that there exists a basis of V over F such that the matrix of T in this basis is of the form

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\gamma_0 & -\gamma_1 & -\gamma_2 & \cdots & -\gamma_{r-1} \end{bmatrix}$$

(7+7)

6. a) State and prove Bessel's inequality.
b) Using Gram-Schmidt process, find the orthogonal and then orthonormal basis from the basis $\left\{\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$.
c) If A is a symmetric matrix then prove that any two eigen vectors from different eigen spaces are orthogonal. **(4+5+5)**

7. a) Define quadratic form and explain the classification of quadratic form with examples.
b) Find the extreme values of $Q(x) = 16x_1^2 + 19x_2^2 + 4x_3^2 + 10x_4^2$ subject to the constraint $x^T x = 1$.
b) Find the singular value decomposition of the matrix $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$.
(4+3+7)

8. a) State and prove Sylvester's law of inertia.
b) Let V be a finite dimensional vector space over F with bases $\beta = \{x_1, x_2, \cdots, x_n\}$ and $\gamma = \{y_1, y_2, \cdots, y_n\}$ and Q be the change of coordinate matrix from γ to β then prove that Ψ_γ and Ψ_β are congruent.
c) Define rank and signature. Find the rank and signature of the quadratic form $x_1^2 - 4x_1x_2 + x_2^2$. **(6+5+3)**

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